48

Chapter

Competitive Strategies (Games Theory)

Chapter Outcomes

After completing the chapter, you will be able to:

- Define a competitive game.
- Describe characteristics of a competitive game.
- Define two person Zero sum Game.
- Apply maximin and minimax principle to solve the game by Saddle point.
- Use algebraic method to solve 2 × 2 game.
- Reduce the size of the game by applying dominance properties.
- Apply graphical method to solve the game.
- Formulate games theory problem as LPP problem and solve the problem as LPP.

48.1 INTRODUCTION

Von Nuemann developed the theory of games in 1928 and he is considered as father of Games theory. Games theory is a type of decision theory in which one's choice is determined after considering possible alternatives available to the opponent playing the game. In ordinary decisions under uncertainty, the decision maker is faced with only a random process. The decision maker has not only to analyse alternative course of action available to him but also consider the possible goals, strategies and choices of the competitor.

Thus, many practical problems require decision making in a competitive situation where there are two or more opposing parties with conflicting interests and where the action depends upon the one taken by the opponent *e.g.* candidates contesting for an election, advertising and promotional campaigns by competing business firms etc. In a competitive situation, with a finite number of competitors is called "a competitive game". The games where there are number of competitive actions (or choices) usually referred to as games of strategy because a person playing one of the these games is merely betting against the odds.

48.2 COMPETITIVE GAMES

The games with the following characters are referred to as a competitive game.

1. There are finite number of players. The number of participants is greater than or equal to two ($n \geq 2$).
2. Players must have conflicting interests.
3. Each player has a finite number of possible courses of action.
4. A play is said to occur when each player chooses one of his course of action.
5. The pay-off's of the game are affected by the selection of the strategies by other players.
6. All the players are aware of the rules governing the choice of action. Each player selects one strategy. The selections are assumed to be made simultaneously and hence no one knows his opponent's strategy.
7. Each pay-off of the game is represented by a single pay-off number representing gain or loss.

48.3 TWO PERSON ZERO SUM GAME

In a game, if the algebraic sum of gains and losses of all the players is called "Zero sum game". A game with two players, where a gain of one player equals the loss to the other is known "two person zero sum game".

The characteristics of two person zero sum game are:

1. There are only two players with exactly the opposite interests.
2. Each player has finite number of strategies.
3. Each specific strategy results in a pay-off.
4. Sum of pay-offs at the end of each play is zero.

48.4 BASIC TERMINOLOGY

(*a*) **Player***:* Each participant (interested party) is called a player.

(*b*) **Play:** A play of the game occurs when each player has chosen a course of action.

(*c*) **Strategy:** The strategy of a player is the predetermined rule by which a player decides his course of action from his own courses of action during the game.

4. **Pure Strategy:** A pure strategy is a decision, in advance of all plays always to choose a particular course of action.
5. **Mixed Strategy:** A mixed strategy is a decision in advance of all plays, to choose a course of action for each pay in accordance with some particular probability distribution.
6. **Pay-off:** Pay-off is outcome of playing the game. A pay-off matrix (gain or loss) is a Table showing the amounts received by the player after all possible plays of the game.

48.5 MAXIMIN-MINIMAX PRINCIPLE

Consider a two person zero sum game involving the set of pure strategies A_1, A_2 , A_3 for A and B_1, B_2 and B_3 for player B and having the following pay-off matrix for the player A.

Player B

		B_1	B_2	B_3	Row minimum
	A_1	9	2	3	2
Player A	A_2	8	$\boxed{6}$	7	6
	A_3	6	4	5	4
Column maximum		9	$\boxed{6}$	7	

Saddle point

Suppose the players adopt an over cautious approach, and if they do so, then, they will assume the worst and act accordingly. Suppose if player A adopts a pessimistic approach to the pay-off matrix, then the worst outcome of A for his strategy A_1 will be 2. (A player A's minimum gain).

Similarly, the minimum outcome for player A's strategy A_2 and A_3 will be 6 and 4 respectively.

A is a maximising player and he aims at maximising his minimum gains. Thus, the maximum of all the minimum values is 6. So the best strategy for A is A_2 which maximises his minimum gains. Thus, maximin refers to maximising minimum gains. Player B is a minimising player and he tries to minimise the maximum gains of player A.

The maximum pay-off of the player A corresponding to the B's strategies B_1, B_2 and B_3 are [9, 6, 7]. Thus, B wants to minimise these maximum gains. The minimum of maximum value of pay-off is $\boxed{6}$. So, player 'B' always tries to use his strategy B_2.

Thus, minimax refers to minimising maximum gains of A.

Saddle Point

Saddle point in a pay-off matrix is the position in the matrix where the maximum of row minimum (maximin) coincides with the minimum of column maximum (minimax). The pay-off at the saddle point is called the value of the game which equals minimax of miximin value.

The optimum solution can be had for the game by applying minimax and maximin.

Thus, the solution of the game is:

Best strategy for A is A_2; Best strategy for B is B_2; Value of the game is 6.

Steps to find out saddle point

1. Find out the minimum element of each row of the pay-off matrix and circle the maximum value.
2. Find out the maximum element in each column of the pay-off matrix and circle the maximum value [☐].
3. If these two values are same (*i.e.* the pay-off containing both '○' and ☐ marks, is the saddle point of the matrix.

Problem: *Solve the game where the pay-off matrix is given by*

Player B

		B_1	B_2	B_3
Player A	A_1	1	3	1
	A_2	0	– 4	–3
	A_3	1	5	–1

Solution: Mark row minimum and column maximum for each row and column.

Player B

Row minimum

		B_1	B_2	B_3
Player A	A_1	$\boxed{1}$	3	$\boxed{1}$
	A_2	0	– 4	–3
	A_3	1	5	–1
Column Maximum		$\boxed{1}$	5	$\boxed{1}$

The row minimum values are [1, – 4, –1]

The column maximum values are [1, 5, 1]

The maximum of minimum value = 1

The minimum of maximum value = 1

∴ Maximin value = Minimax value

Saddle point exists and this pay-off of (1) occurs at two positions as shown in table above.

The solution to the game is

Best strategy for $A = A_1$

Best strategy for $B = B_1$ or B_3

Value of the game = 1 (pay-off at saddle point)

Problem: *The pay-off matrix of a game is given below. Find the solution of the game to a and B.*

Player B

		B_1	B_2	B_3	B_4	B_5	Row min	
	A_1	–2	0	0	5	3	–2	
Player A	A_2	4	2	$\boxed{1}$	2	5	1	Maximin value
	A_3	– 4	–3	0	–3	6	– 4	
	A_4	5	1	–5	2	– 6	6	
	Max value	5	2	$\boxed{1}$	5	6		

Solution: First, find out the saddle point;

The minimum of each row is [–2, 1, – 4, – 6]; Maximum of minimum value of the row = 1

The maximum of each column = [5, 2, 1, 5, 6]; Minimax value = 1

The saddle point pay-off = 1

∴ Solution of the game is

(*i*) the best strategy for $A = A_2$

(*ii*) the best strategy for $B = B_3$

(*iii*) value of the game = 1

48.6 TWO PERSON ZERO SUM GAME WITH MIXED STRATEGIES (WITHOUT SADDLE POINT).

For any two person zero sum game where the optimal strategies are not pure strategies and having the pay-off matrix for player 'A' as

		Player B	
		B_1	B_2
Player A	A_1	a_{11}	a_{12}
	A_2	a_{21}	a_{22}

Let x_1 and x_2 be the probabilities of selecting the strategies A_1 and A_2 by player 'A'. (here $x_2 = (1 - x_1)$ and y_1 and y_2 be the probabilities of selecting strategies B_1 & B_2 by player B $(y_2 = (1 - y_1)$.

The optimal strategies for the players are:

For the player A, $$x_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

$$x_2 = (1 - x_1) = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

For the player B, $$y_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

$$y_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

Value of the game (V) $$= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}$$

(Note that the denominator in all the formulas are the same).

Problem: *For the pay-off matrix given below, find the optimal strategies for the players and value of the game.*

		Player B		
		B_1	B_2	
Player A	A_1	8	–3	(x_1)
	A_2	–3	1	(x_2)
		(y_1)	(y_2)	

Solution: In the above problem, saddle point does not exist. So, it is mixed strategy problem. Let x_1 and x_2 represent the probabilities with which player A will use his strategies A_1 and A_2 and y_1 and y_2 be the probabilities with which the player B chooses his strategy B_1 and B_2. The optimum mixed strategies are:

$$x_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15}; \ x_2 = (1 - x_1) = \frac{11}{15}$$

For the player 'B', the probability with which he uses strategies B_1 and B_2 are:

$$y_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15};\ y_2 = (1-y_1) = \frac{11}{15}$$

The expected value of the game is

$$V = \frac{8-(-3)(-3)}{8+1-(-3-3)} = -\frac{1}{15}$$

48.7 DOMINANCE PROPERTY

Principle of dominance is applicable to both pure strategies and mixed strategies. Sometimes, it is observed that one of the pure strategies of either players is always inferior to at least one of the remaining strategies. The superior strategies are said to dominate the inferior ones. The player would have no incentive to use inferior strategies which are dominated by the superior ones. In such cases of dominance, the size of the pay-off matrix by deleting those strategies which are dominated by the others.

The dominance properties are:

1. If all the elements of a row say K^{th}, are less than or equal to the corresponding elements of any other row (say r^{th} row), then k^{th} row is dominated by the r^{th} row.
2. If all the elements of column, say k^{th}, are greater than or equal to the corresponding elements of any other column, say r^{th} then K^{th} column is dominated by the r^{th} row.
3. A pure strategy may be dominated, if it is inferior to average of two or more other pure strategies.

Problem: *The pay-off matrix of the player 'A' is given.*

Player *B*

Player *A*		I	II	III	IV	V
	I	2	4	8	3	4
	II	5	6	7	3	8
	III	6	7	8	9	7
	IV	4	2	4	8	3

Using dominance property obtain the optimum strategies for both the players and determine the value of the game.

Solution: As all the elements of column I are either less or equal (≤) to respective elements of column III, column I is said to have dominance over column III and column III can be eliminated. Hence, we get the pay-off matrix as shown.

Player *B*

Player *A*		I	II	IV	V
	I	2	4	3	4
	II	5	6	3	8
	III	6	7	9	7
	IV	4	2	8	3

As all the elements of row III are greater than the respective elements of row IV, row III is said to have dominance over row IV, and hence, the IV can be eliminated. Hence, we get

Player B

		I	II	IV	V
	I	2	4	3	4
Player A	II	5	6	3	8
	III	6	7	9	7

As all the elements of column I are less than the respective elements of column V, column I is said to dominate column V and hence, column V is eliminated.

Player B

		I	II	IV
	I	2	4	3
Player A	II	5	6	3
	III	6	7	9

All the elements of row III are greater than the respective elements of row I and II and now row III is said to have dominance over row I and row II and hence, eliminating row I and row II, we get

Player B

Player A		I	II	III
	III	6	7	9

Now, the minimum value of row III is 6 and hence it is the saddle point

Hence, the best strategy for A is III; the best strategy for B is I; and value of the game is 6.

Problem: *Solve the following game*

Player B

		1	2	3	4
	1	3	2	4	0
	2	3	4	2	4
Player A	3	4	2	4	0
	4	0	4	0	8

Solution: Since all the elements of 3rd row are greater than or equal to the corresponding elements of 1st row, third (3rd) row is dominating 1st row and hence row 1 can be eliminated.

Player B

		1	2	3	4
	1	3	4	2	4
Player A	3	4	2	4	0
	4	0	4	0	8

Again, all the elements of the 1^{st} column are greater than or equal to the corresponding elements of the 3^{rd} column.

$\therefore$ 3^{rd} column is dominating the first column. Eliminating 1^{st} column, the matrix is reduced to

Player *B*

		2	3	4
	2	4	2	4
Player *A*	3	2	4	0
	4	4	0	8

Here, the linear combination of 2^{nd} and 4^{th} column dominates the 2^{nd}, because

$$4 > \frac{2+4}{2},\ 2 = \frac{4+0}{2} \text{ and } 4 = \frac{0+8}{2}$$

Eliminating 1^{st} column, the reduced matrix becomes,

Player *B*

		3	4
	2	2	4
Player *A*	3	4	0
	4	0	8

Here again, the convex linear combination of 3 and 4 of player '*A*' dominate 2^{nd} because

$$2 = \frac{4+0}{2},\quad 4 = \frac{0+8}{2}$$

$\therefore$ Eliminating 1^{st} row, the reduced matrix becomes

Player *B*

		3	4
Player *A*	3	4	0
	4	0	8

The 2×2 matrix can be solved algebraically as follows:

Let x_1 and x_2 be the probabilities of mixed strategies for player '*A*' and y_1, and y_2 be the probabilities of mixed strategies for player '*B*'.

$\therefore$ $$x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} + a_{22} - a_{21}} = \frac{8-0}{4-0+8-0} = \frac{8}{12} = \frac{2}{3}$$

and $$x_2 = (1 - x_1) = 1 - \frac{2}{3} = \frac{1}{3}$$

and $$y_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} + a_{22} - a_{21}} = \frac{8-0}{4-0+8-0} = \frac{8}{12} = \frac{2}{3}$$

$$\therefore \qquad y_2 = (1 - y_1) = \left(1 - \frac{2}{3}\right) = \frac{1}{3}$$

Thus, optimum strategies are:

For player A, [0, 0, 2/3, 1/3]; For player B, [0, 0, 2/3, 1/3]

and the value of the game is

$$\frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{4 \times 8 - 0 \times 0}{4 + 8 - (0 \times 0)} = \frac{32}{12} = \boxed{\frac{8}{3}}$$

Problem: *Solve the following game by using the principle of dominance*

Player B

		B_1	B_2	B_3	B_4	B_5	B_6
	A_1	4	2	0	2	1	1
	A_2	4	3	1	3	2	2
Player A	A_3	4	3	7	–5	1	2
	A_4	4	3	4	–1	2	2
	A_5	4	3	3	–2	2	2

Solution: The pay-off matrix has no saddle point.

All the elements of row A_1 are dominated by row A_2 and row A_5 is dominated by row A_4. Hence row A_1 and A_5 can be eliminated.

Hence, the pay-off matrix is reduced to

Player B

		B_1	B_2	B_3	B_4	B_5	B_6
	A_2	4	3	1	3	2	2
Player A	A_3	4	3	7	–5	1	2
	A_4	4	3	4	–1	2	2

From player B's point of view, column B_1 and B_2 are dominated by columns B_4, B_4 and B_6 and column B_6 is dominated by column B_5. Hence strategies B_1, B_2 and B_6 are eliminated. The modified pay-off matrix is

Player B

		B_3	B_4	B_5
	A_2	1	3	3
Player A	A_3	7	–5	1
	A_4	4	–1	2

Now, none of the single row or column dominates another row or column *i.e.* none of the pure strategies of A and B is inferior to any of the other strategies.

However, column B_5 is dominated by the average of column B_3 and B_4.

$$\left\{\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right\} \text{ or } [2, 1, 3/2]$$

This strategy of B is superior to strategy B_5 because the B_5 strategy will result him in greater loss. So, the strategy B_5 can be eliminated. The modified matrix is

Player B

		B_3	B_4
	A_2	1	3
Player A	A_3	7	–5
	A_4	4	–1

The average of A's pure strategies A_2 and A_3 $\left\{\frac{1+7}{2}, \frac{3-5}{2}\right\}$ or (4, –1) which is obviously same as strategy A_4. Thus, strategy A_4 can be eliminated.

Thus we get the resulting matrix (2 × 2) is given by

Player B

		B_3	B_4
	A_2	1	3
Player A	A_3	7	–5

Solving this (2 × 2) game by arithmetic method, the solution to the game is

Optimal strategy for A – [0, 6/7, 1/7, 0, 0] Optimal strategy for B [0, 0, 4/7, 3/7, 0, 0]

The value of the game to player A is $\frac{13}{7}$.

48.8 GRAPHICAL METHOD FOR SOLVING 2 × N OR M × 2 GAMES

Graphical method is applicable to only those games in which one of the players has two strategies only. The advantage of this method is that is that it solves the problem relatively faster.

The graphical method consists of two graphs.

(*i*) the pay-off (gains) available to player 'A' against his strategies and options.

(*ii*) the pay-off (losses) faced by player 'B' against his strategies and options.

Illustration of the graphical method

The pay-off matrix is given below.

Player B

		1	2	3
Player A	1	4	2	3
	2	2	5	6

The problem has no saddle point.

∴ There are no pure strategies and mixed strategies are to be adopted.

Player 'A' adopts the probabilities x_1 and x_2 for strategies 1 and 2. for player 'B' the mixed strategies with probabilities are y_1, y_2 and y_3 respectively. At the optimal level with the value of the game, as V the following relationship can be established.

$$x_1 + x_2 = 1 \quad ...(1)$$

$$y_1 + y_2 + y_3 = 1 \quad ...(2)$$

$$4x_1 + 2x_2 \geq V \quad ...(3)$$

$$2x_1 + 5x_2 \geq V \quad ...(4)$$
$$3x_1 + 6x_2 \geq V \quad ...(5)$$
$$4y_1 + 5y_2 + 6y_3 \leq V \quad ...(6)$$
$$2y_1 + 5y_2 + 6y_3 \leq V \quad ...(7)$$

The above equations can be written in terms of the player having two strategies. *i.e.* in terms of player *A*.

$\therefore \qquad x_2 = 1 - x_1$

Substituting the value of x_2 in equations (3), (4) and (5). we get, Equation 93) can be written as:

$$4x_1 + 2(1 - x_1) \geq V$$

i.e. $\qquad 2x_1 + 2 \geq V$

i.e. $\qquad V - 2x_1 \leq 2 \quad ...(8)$

Equation (4) can be written as:

$$2x_1 + 5(1 - x_1) \geq V$$

i.e. $\qquad 3x_1 + 5 \geq V$

i.e. $\qquad V + 3x_1 \leq 5 \quad ...(9)$

Equation (5) can be written as:

$$3x_1 + 6(1 - x_2) \geq V$$

i.e. $\qquad -3x_1 + 6 \geq V$

$$V + 3x_1 \leq 6 \quad ...(10)$$

Player *A*'s objective is to maximise the value of '*V*' and to find the combination of x_1 and x_2 which gives the maximum value.

The graph of x_1 versus *V* can be drawn with the relationships in equations (8), (9) and (10) by plotting x_1 on *x*-axis and '*V*' on y-axis. The range of x_1 is between 0 and 1, and so we plot the graph within 0 and 1 of x_1.

Equation (8) gives,

When $\qquad x_1 = 0, V = 2$

$\qquad x_1 = 1, V = 4.$

From equation (9), when $\qquad x_1 = 0, V = 5$

$\qquad x_1 = 1, V = 2$

From equation (10), when

$\qquad x_1 = 0, V = 6$

$\qquad x_1 = 1, V = 3$

Plotting these equations on the graph, as shown in Fig. 48.1 the feasible region is given by the area below *AGD*.

The maximum value of '*V*' in this region is given at point *G*. At this point, $V = 3\frac{1}{5}$ and

$$x_1 = \frac{3}{5},\ x_2 = 1 - x_1 = 2/5$$

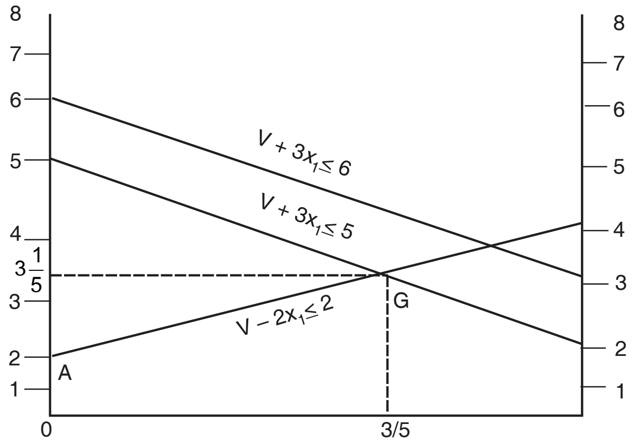

Fig. 48.1: Graph of x_1 Versus 'V'

This diagram reveals that B will never play $y_3$in which case his loss will be more than what it is by playing with strategy y_2 as $V + 3x_1 \leq 6$ (representing strategy y_3) which is above $V + 3x_1 \leq 5$ (representing strategy y_2). Eliminating this strategy for player B, we can plot the graphs for player B.

Equations (2), (6) and (7) can be rewritten as

$$y_1 + y_2 = 1 \quad ...(11)$$

$$4y_1 + 2y_2 \leq V \quad ...(12)$$

$$2y_1 + 5y_2 \leq V \quad ...(13)$$

$\therefore \quad y_2 = 1 - y_1$ Substituting for y_2 in the equations (12) becomes

$4y_1 + 2(1 - y_1) \leq v$ *i.e.* $2y_1 + 2 \leq V$

i.e. $\quad V - 2y_1 \geq 2 \quad ...(14)$

Equation (13) becomes

$$2y_1 + 5(1 - y_1) \leq V, \text{ i.e. } -3y_1 + 5 \leq V$$

i.e. $\quad V + 3y_1 \geq 5 \quad ...(15)$

From equation (14), when $y_1 = 0, V = 2$

$y_1 = 1, V = 4$

From equation (15), when $y_1 = 0, V = 5$

$y_1 = 1, V = 2$

Plotting these graphs, we have feasible region above CEB as shown in Fig. 48.2.

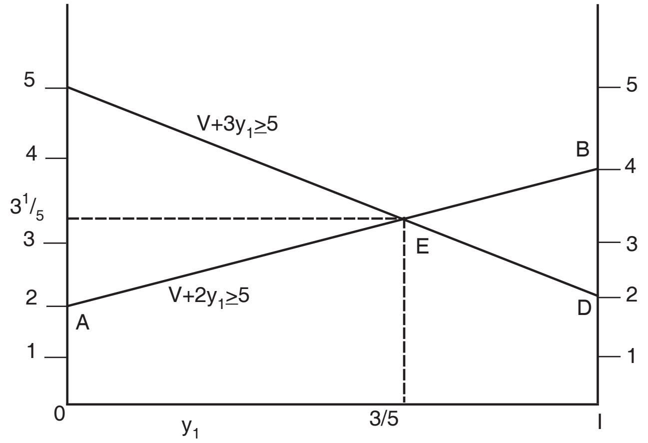

Fig. 48.2: Plot of y_1 Versus V

B's objective is to minimise the value of V, this arises in the feasible region at point E. Giving the values as:

$$V = 3\frac{1}{5}, y_1 = \frac{3}{5}, y_2 = 1 - y_1 = \frac{2}{5},$$

48.9 SOLUTION OF GAME BY LINEAR PROGRAMMING APPROACH

Game theory has a strong relationship with linear programming since every finite two person zero sum game can be expressed as a linear programme and conversely every linear programme can be represented as a game.

The linear programming approach for the game can be represented as follows:

Let player 'A' has 'm' course of action (A_1, A_2, A_3 ..... A_m) and player 'B' has 'n' courses of action (B_1, B_2, B_3 ......B_n) The pay-off to player 'A' if he selects strategy A_i and player 'B' selects strategy Bj is a_{ij}.

Mixed strategy for player 'A' is defined by the probabilities p_1, p_2, p_3 ..... p_m where $p_1 + p_2 + p_m = 1$.

and mixed strategy for player 'B' is defined by the probabilities q_1, q_2, ... q_n where $q_1 + q_2 + q_3 + ... + qn = 1$ let 'V' be value of game to A. Then, the game can be defined as a linear programming problems as follows:

Determine the unknown quantities p_1, p_2, ... p_n with an objective is to maximise the value of the game 'V' such that the following constraints are satisfied.

$$\begin{aligned}
a_{11}p_1 + a_{21}p_2 + \ldots\ldots + a_{m1}p_m &\geq V \\
a_{12}p_1 + a_{22}p_2 + \ldots\ldots + a_{m2}p_m &\geq V \\
\vdots \qquad\qquad & \vdots \\
a_{1n}p_1 + a_2np_2 + \ldots\ldots + a_{mn}p_m &\geq V \\
p_1 + p_2 + p_3 + \ldots p_m &= 1 \\
p_1, p_2, p_3 \ldots p_m &\geq 0
\end{aligned}$$

If the pay-off's *i.e.* a_{ij} are negative, which is not allowed in linear programming simplex method. To prevent this, a constant 'K' can be added to all the elements of the pay-off matrix to make them all non-negative. Once the optimal solution is obtained for L.P.P., the constant 'k' can be subtracted from the objective value to get the true value of the game.

Now, dividing each inequality by 'V', we get

$$\begin{aligned}
&a_{11}(p_1/V) + a_{21}(p_2/V) + \cdots + a_{m1}(p_m/V) \geq 1 \\
&a_{12}(p_1/V) + a_{22}(p_2/V) + \cdots + a_{m2}(p_m/V) \geq 1 \\
&\quad \vdots \\
&a_{1n}(p_1/V) + a_{2n}(p_2/V) + \cdots + a_{mn}(p_m/V) \geq 1 \\
&(p_1/V) + (p_2/V) + p_3/V + \cdots + p_m/V \geq 1/V
\end{aligned}$$

To simplify the problem, we define new variables.

Let $p_1/V = x_1,\ p_2/V = x_2,\ .....\ p_n/V = x_n$

The objective of the firm is to maximise the value of V, which is equivalent to minimising $1/V$.

The resulting linear programming problem can be written as,

Minimize $Z = \dfrac{1}{V}$ or $Z = x_1 + x_2 + \cdots + x_n$

Subjected to constraints,

$$\begin{aligned}
&a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_n \geq 1 \\
&a_{12}x_2 + a_{22}x_2 + \cdots + a_{m2}x_n \geq 1 \\
&\quad \vdots \\
&a_{1n}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n \geq 1 \\
&x_1, x_2 \ldots\ldots x_n \geq 0
\end{aligned}$$

Example: *Express the following game as a Linear Programming Problem.*

	Player B		
	3	–4	2
Player A	1	–3	–7
	–2	4	7

Solution: Let x_1 , x_2 , x_3 and y_1 , y_2 and y_3 be the probabilities by which players A and B select their strategies respectively.

The game has no saddle point.

The value of the game lies between –2 and +3

[Maximin = 2 and Minimax = 3]

Since the value of the game lies between –2 and +3, it is possible that the value of the game (V) may be negative or zero because $-2 \le V \le 3$.

Thus, a constant 'K' is added to all the elements of the matrix which is at least equal to the negative of the maximin value *i.e.* 'K' must be ≥ 2. Assume the value of $K = 3$.

The given matrix is thus modified as follows:

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{array}{|ccc|} \hline 6 & -1 & 5 \\ 4 & 0 & -4 \\ +1 & 7 & 10 \\ \hline \end{array} \end{array}$$

Let the value of the game to A is V.

Let the strategies of player A is designated by p_1, p_2, p_3 respectively.

Then, the constraints are

$$6p_1 + 4p_2 + p_3 \ge V$$
$$-p_1 + 7p_3 \ge V$$
$$5p_1 - 4p_2 + 10p_3 \ge V$$

Let $x_i = \dfrac{pi}{V}$ $(i = 1, 2, 3.....)$

Then, the problem for 'A' is

Minimise $x_1 + x_2 + x_3$ Subjected to constraints

$$6x_1 + 4x_3 + x_3 \ge 1 \quad ...(1)$$
$$-x_1 + 7x_3 \ge 1 \quad ...(2)$$
$$5x_1 - 4x_2 + 10x_3 \ge 1 \quad ...(3)$$
$$x_1 , x_2 , x_3 \ge 1$$

From B's point of view, it can be expressed as:

Maximise $y_1 + y_2 + y_3$ Subjected to constraints

$$6y_1 - y_2 + 5y_3 \le 1$$
$$4y_1 - 5y_3 \le 1$$
$$y_1 + 7y_2 + 10y_3 \le 1$$
$$y_1 , y_2 , y_3 \ge 0$$

The problem can now be solved using simplex procedure.

48.10 GAME WITH MIXED STRATEGIES

If a game has no saddle point, then the game is said to have mixed strategies.

To demonstrate the determination of mixed strategies, consider the 2 × 2 payoff matrix with respect to Player A which has no saddle point (Table 48.1).

Table 48.1: A Game with mixed strategies

		B: 1	B: 2
A	1	a	b
	2	c	d

Algorithm to determine mixed strategies

Step 1: Find the absolute value of $a - b$ (*i.e.* $|a - b|$) and write it against Row 2.

Step 2: Find the absolute value of $c - d$ (*i.e.* $|c - d|$) and write it against Row 1.

Step 3: Find the absolute value of $a - c$ (*i.e.* $|a - c|$) and write it against Column 2.

Step 4: Find the absolute value of $b - d$ (*i.e.* $|b - d|$) and write it against Column 1.

The results of the above steps are summarized in Table 12.5. The absolute values are called as oddments.

Table 48.2: Payoff matrix with oddments

		B: 1	B: 2	Oddments
A	1	a	b	$\|c - d\|$
	2	c	d	$\|a - b\|$
	Oddments	$\|b - d\|$	$\|a - c\|$	

Step 5: Compute the probabilities of selection of the alternatives of Player A (p_1 and p_2) and that of Player B (q_1 and q_2).

$$p_1 = \frac{|c-d|}{|a-b|+|c-d|}$$

$$p_2 = \frac{|a-b|}{|a-b|+|c-d|}$$

$$q_1 = \frac{|b-d|}{|a-c|+|b-d|}$$

$$q_2 = \frac{|a-c|}{|a-c|+|b-d|}$$

The value of the game can be computed using any one of the following formulae:

$$V = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|}$$

$$= \frac{b|c-d|+d|a-b|}{|a-b|+|c-d|}$$

$$= \frac{a|b-d|+b|a-c|}{|a-c|+|b-d|}$$

$$= \frac{c|b-d|+d|a-c|}{|a-c|+|b-d|}$$

Example: *Consider payoff matrix with respect of Player A and solve it optimally:*

		B: 1	B: 2
A	1	6	9
	2	8	4

Solution: The maximum and minimax values of the given problem are shown in below:

		B: 1	B: 2	Maximin	
A	1	6	9	6	(maximum)
	2	8	4	4	
		8 (minimax)	9		

In this problem, the maximin value (6) is not equal to the minimax value (8). Hence, the game has no saddle point. Under this situation, the formulae given in this section are to be used to find the mixed strategies of the players and also the value of the game. The computations of oddments of the game are sumarized in Table below:

Table 48.3: Payoff matrix with Oddments

		B: 1	B: 2	Oddments
A	1	6	9	4
	2	8	4	3
Oddments		5	2	

Let p_1 and p_2 be the probabilities of selection Alternative 1 and of Alternative 2, respectively of Player A. Also q_1 and q_2 be the probability of selection of Alternative 1 and of Alternative 2 of Player B, respectively. Then, we have

$$p_1 = \frac{|c-d|}{|a-b|+|c-d|} = \frac{4}{3+4} = \frac{4}{7}$$

$$p_2 = \frac{|a-b|}{|a-b|+|c-d|} = \frac{3}{3+4} = \frac{3}{7}$$

$$q_1 = \frac{|b-d|}{|a-c|+|b-d|} = \frac{5}{2+4} = \frac{5}{7}$$

$$q_2 = \frac{|a-c|}{|a-c|+|b-d|} = \frac{2}{2+5} = \frac{2}{7}$$

where, the value of the game is

$$V = \frac{a|c-d|+c|a-b|}{|a-b|+|c-d|} = \frac{(6)(4)+(8)(3)}{4+3} = \frac{48}{7}$$

Hence, the strategies of Player A is: $A(4/7, 3/7)$ and of Player B: $B(5/7, 2/7)$

The value of the game $= \frac{48}{7} = 6\frac{6}{7}$

SUMMARY

Games or 'Strategic Interactions' can be found in all walks of life. Examples of such scenarios are two firms competing for market share, politicians contesting elections, different bidders participating in an auction for wireless spectrum, coal blocks etc. Game theory provides a convenient framework to model and interpret the behavior of participants in such strategic interactions. Hence it can be applied to solve a wide variety of problems involving diverse areas such as Markets, Auctions, Online Retail, Cold War, Paying Taxes, Bargaining, Elections, Portfolio Management etc.

Developed in 1950s by mathematicians John von Neumann and economist Oskar orgenstern. Games Theory Designed to evaluate situations where individuals and organizations can have conflicting objectives. A game is said to be ***zero-sum*** if wealth is neither created nor destroyed among the players. A game is said to be ***non-zero-sum*** if wealth may be created or destroyed among the players (*i.e.* the total wealth can increase or decrease).

Game theory is the study of how people behave in strategic situations. Strategic decisions are those in which each person, in deciding what actions to take, must consider how others might respond to that action. Four elements to describe a game:

- Players: Participants
- rules: when each player moves, what are the possible moves, what is known to each player before moving;
- outcomes of the moves;
- payoffs of each possible outcome: how much money each player receive for any specific outcome.

A ***strategy*** is a plan of action by which a player has a decision rule to determine their set of moves for every possible situation in a game. A strategy is said to be ***pure*** if it at every stage in the game it specifies a particular move with complete certainty. A strategy is said to be ***mixed*** if it applies some randomisation to at least one of the moves.

For each game, there are typically multiple pure strategies.

Dominant strategy: One firm's best strategy may not depend on the choice made by the other participants in the game. Leads to Nash equilibrium because the player will use the dominant strategy and the other will respond with its best alternative.

Dominated strategies: An alternative that yields a lower payoff than some other strategies a strategy is dominated if it is always better to play some other strategy, regardless of what opponents may do. It simplifies the game because they are options available to players which may be safely discarded as a result of being strictly inferior to other options.

There are many solution methods to solve the games

1. Saddle point method
2. Games without saddle point Algebraic method, Oddment method
3. Dominance properties to reduce size of the game
4. Graphical method
5. Solution of games using LPP

REFERENCES FOR FURTHER READING

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REVIEW QUESTIONS

1. Explain the following terms with reference to Games theory
   (*i*) Competitive game
   (*ii*) Two person Zero sum game
   (*iii*) Pay off and Pay off matrix
   (*iv*) Saddle point and Value of the game
   (*v*) Pure and Mixed strategies
   (*vi*) Dominance Property
2. Illustrate with an example how to use dominance properties to reduce the size of rectangular game?
3. Formulate games theory problem as an LPP problem
4. What are the limitations of games theory?
5. What are the solution methods available for games without saddle points?

PROBLEMS

1. Two companies manufacturing air conditioners compete for supplying slit AC's to a new five star hotel to be constructed. Each company has listed its alternatives/strategies for selling AC's. The strategies of Company *A* are as follows; (*i*) giving special discount (*ii*) Giving additional warranty (*iii*) Special gifts. Company *B* the strategies are (*i*) Special price offer (*ii*) 20% additional discount on other products (*iii*) Fre training to users. The estimated gains and losses in lakhs of rupees for company *A* for various combination of alternatives of both companies are summarized in the table below. Find the optimum strategies for or companies?

B

A		1	2	3
	I	40	45	50
	II	20	45	60
	III	25	30	30

2. Solve the following Game using dominance rules

Player B

Player A		1	2	3	4	5
	1	6	15	30	21	6
	2	3	3	6	6	4
	3	12	12	24	36	3

3. Solve the following game by Graphical Method

B

		1	2	3	4	5
A	1	4	2	5	–6	6
	2	7	–9	7	4	8

4. Consider the payoff matrix for player *A* and determine optimum strateguies using Graphical method.

B

		1	2
A	1	5	3
	2	6	4
	3	2	–7
	4	9	–8

5. The labor union and management of a particular company have been negotiating a new labor contract. However, negotiations have now come to an impasse, with management making a "final" offer of a wage increase of 80 per hour and the union making a "final" demand of a 120 per hour increase. Therefore, both sides have agreed to let an impartial arbitrator set the wage increase somewhere between 80 and 120 ₹ per hour (inclusively).The arbitrator has asked each side to submit to her a confidential proposal for a fair and economically reasonable wage increase (rounded to the nearest dime). From past experience, both sides know that this arbitrator normally accepts the proposal of the side that gives the most from its final figure. If neither side changes its final figure, or if they both give in the same amount, then the arbitrator normally compromises halfway between 100 ₹ in this case). Each side now needs to determine what wage increase to propose for its own maximum advantage. Formulate this problem as a two-person, zero-sum game.

6. Two manufacturers currently are competing for sales in two different but equally profitable product lines. In both cases the sales volume for manufacturer 2 is three times as large as that for manufacturer 1. Because of a recent technological breakthrough, both manufacturers will be making a major improvement in both products. However, they are uncertain as to what development and marketing strategy to follow. If both product improvements are developed simultaneously, either manufacturer can have them ready for sale in 12 months. Another alternative is to have a "crash program" to develop only one product first to try to get it marketed ahead of the competition. By doing this, manufacturer 2 could have one product ready for sale in 9 months, whereas manufacturer 1 would require 10 months (because of previous commitments for its production facilities). For either manufacturer, the second product could then be ready for sale in an additional 9 months.

For either product line, if both manufacturers market their improved models simultaneously, it is estimated that manufacturer 1 would increase its share of the total future sales of this product by 8 percent of the total (from 25 to 33 percent). Similarly, manufacturer 1 would increase its share by 20, 30, and 40 percent of the total if it marketed the product sooner than manufacturer 2 by 2, 6, and 8 months, respectively. On the other hand, manufacturer 1 would lose 4, 10, 12, and 14 percent of the total if manufacturer 2 marketed it sooner by 1, 3, 7, and 10 months, respectively. Formulate this problem as a two-person, zero-sum game, and then determine which strategy the respective manufacturers should use according to the minimax criterion.